Appendix B

Dynamic Performance Changes Produced by Numerical Integration Algorithms

This appendix relates to Chapter 3, §3.1.2.

The dynamic characteristics of a discrete-time system can be analyzed and compared with those of the corresponding continuous-time system by the use of Laplace transformed equations and comparisons between the related transfer functions.

As demonstrated in the following, the proposed techniques [3-9] consider, for each numerical integration method, an “equivalent integrator” characterized by the following generic transfer function: Q(τs)/s, with τ the integration step length. This integrator substitutes the classical 1/s operator of the ideal integrator.

Fundamentals. Passing from a system description based on continuous-time differential equations to finite-differences algebraic equations, the dynamic response can change. With reference to a linear, constant parameter dynamic system, the continuous-time model can be described by:

$$\begin{aligned} \dot{x}(t) &= A \cdot x(t) + B \cdot u(t) \\ y(t) &= C \cdot x(t) \end{aligned} \qquad (1)$$

where x(t), y(t), u(t) are respectively the state, output and input system vectors variables and A, B, C are constant coefficients matrices.

Through a variable transformation, the A matrix can be reduced to the corresponding diagonal matrix Λ, with diagonal coefficients (eigenvalues) λ_1. In fact, starting from the variables transformation $z(t) = \Gamma^{-1} x(t)$, in which $\Gamma^{-1} A \Gamma = \Lambda$, the equation (1) can be written as (2):

$$\begin{aligned} \dot{z}(t) &= \Lambda \cdot z(t) + W \cdot u(t) \\ y(t) &= H \cdot z(t) \end{aligned} \qquad (2)$$

where: $W = \Gamma^{-1} B$ and $H = C\,\Gamma$.

Applying the Laplace transform, the (2) becomes (3):

$$\begin{aligned} z &= \frac{1}{s} I_n [\Lambda \cdot z + W \cdot u] \\ y &= H \cdot [s \cdot I_n - \Lambda]^{-1} \cdot W \cdot u = F(s) \cdot u \end{aligned} \qquad (3)$$

The corresponding discrete-time model can be expressed through finite-differences algebraic equations, depending on the integration step τ (sample-hold time interval) and the integration algorithm.

There are two basic classes of numerical integration methods having the step integration length τ constant. They are called: "single step" and "multi-step" methods. The single-step integration methods are based on the following generic law:

$$x(t+\tau) = x(t) + \tau\varphi(x,t,\tau)$$

where the function $\varphi(\cdot)$ called "incremental" depends on the integration method and on the considered dynamic system equations (1).

With reference to the Explicit Euler method (EE), the resulting discrete-time model of (1) is (4):

$$\begin{aligned} x(t+\tau) &= x(t) + \tau \cdot [A \cdot x(t) + B \cdot u(t)] \\ y(t+\tau) &= C \cdot x(t+\tau) \end{aligned} \qquad (4)$$

and after the variable transformation becomes (5):

$$\begin{aligned} z(t+\tau) &= z(t) + \tau \cdot [\Lambda \cdot z(t) + W \cdot u(t)] \\ y(t+\tau) &= H \cdot z(t+\tau) \end{aligned} \qquad (5)$$

Applying the Laplace transform,(5) becomes (6):

$$\begin{aligned} z &= \frac{Q(\tau s)}{s} I_n \cdot [\Lambda \cdot z + W \cdot u] \\ y &= H[s \cdot I_n - Q(\tau s) \cdot \Lambda]^{-1} Q(\tau s) \cdot W \cdot u \\ y &= F_1(s) \cdot u \end{aligned} \qquad (6)$$

Passing from (2) to (5) entails a change of corresponding transfer functions, compare (3) and (6). The Laplace operator 1/s, which represents the ideal integrator for the continuous-time model, assumes, for the discrete-time equations, the expression $Q(\tau s)/s$, in agreement with the preliminary hypothesis.

Also note that:

$$\lim_{\tau \to 0} Q(\tau s) = 1 \qquad \text{(compare also equations (3) and (6)).}$$

The more Q(τs) is different from 1, the greater are the altered dynamics going from a continuous to a discrete model. In general, to obtain a negligible alteration of system dynamic behavior, the integration step and therefore the cycle time of digital controller which includes input acquisition, state variable integration and output assignment, must be smaller than the smallest time constant of the regulator (about 20 ms for the AVRs). Furthermore, simple and explicit integration algorithms are preferred to the multi-step ones.

As Λ is a diagonal matrix, equations (3), (6) show an important characteristic of linear system: the altered dynamics of the eigenvalues, due to the conversion from continuous to discrete-time model, are decoupled from each other.

According to the fact that the altered dynamics of each pole depends on the pole itself and on the step τ and the method used for integration, a simplified linear analysis, with first order model, can be considered:

$$F(s) = \frac{c}{s-\lambda}$$

$$F_1(s) = \frac{c \cdot Q(\tau s)}{s - \lambda \cdot Q(\tau s)} = \frac{c + \Delta c}{s - (\lambda + \Delta\lambda)} + \sum_{r=1}^{N-1} \frac{\Delta c_r}{s - \lambda_r}$$

The first term of $F_1(s)$ is related to continuous *F(s)* system, with small changes of the pole and the residue values. The second term shows possible spurious singularities due to numerical integration algorithm (N is the order of the finite-difference equations systems and depends on the employed integration method). The spurious terms exist only for multi-step integration algorithms as: EXTRA, while they disappear in the single step methods, as Eutrap, Eulero, Runge-Kutta, which are normally used in real-time applications.

The Laplace integrator Q(τs)/s for the discrete-time equations. Now the problem is to define, case by case the transfer function Q(τs)/s for computing Δλ and Δ*c*. In the following this problem is solved for the more complex multi-step integration methods, from which it's easy to determine the solution for the single-step method.

In the case of multi-step methods, the integration of a differential equation such as:

$$x(t) = f(x \cdot u)$$

is basically obtained by the use of a numerical algorithm having the following general formulation:

$$x_{k+1} = a_1 x_k + \ldots + a_n x_{k+1-n} + \tau(b_0 f_{k+1} + b_1 f_k + \ldots + b_n f_{k+1-n}) \qquad (7)$$

where:

- n in the order of the considered integration method;

- $x_k \underline{\underline{\Delta}}\, x(k\tau)$; $u_k \underline{\underline{\Delta}}\, u(k\tau)$; $f_k \underline{\underline{\Delta}}\, f(x_k, u_k)$

Note that the general equation (7) can be considered representative of a numerical integration method when:

$$\sum_{k=1}^{n} a_k = 1 \qquad (8)$$

$$\sum_{k=1}^{n} k \cdot a_k = \sum_{k=0}^{n} b_k \qquad (9)$$

In fact the equations (8) and (9) must to be satisfied to obtain the numeric sequence x_k ramping with slope 1 with constant derivative $f_k = 1$.

To define the transfer function associated with the numerical integration algorithm, the first step requires linking to the finite-differences equation (7), the corresponding continuous-time equation (10):

$$x(t) = a_1 x(t-\tau) + \ldots + an \cdot x(t-n\tau) +$$
$$+ \tau\left[b_0 f(t) + b_1 f(t-\tau) + \ldots + b_n f(t-n\tau)\right] \qquad (10)$$

where:

$$f(t) = f(x(t); u(t))$$

Note that equation (10), coincident with equation (7) at the instant $t=t_k=k\tau$, fixes a particular link between the numerical sequence $\{x_k; t_k\}$ and the sequence $\{f_k; t_k\}$, link depending on $u(t)$ shape. In particular, if the $u(t)$ input is piece-wise constant, that is constant value between two subsequent instants: $t_k=k\tau$ and $t_{k+1}=(k+1)\tau$, in that case in equation (10) also the function $f(t)$ and $x(t)$ respectively input and output of the integrator, are piece-wise constant between the instant t_k and t_{k+1}. In this way the physical process is respected by which the analog outputs in real-time applications are constant during the interval t_k and t_{k+1}.

The piece-wise constant input is realistic when produced by a sample and hold mechanism of the real continuous time input (that is when using digital technology).

Taking into account the previous consideration it’s now possible to simply compute the transfer function corresponding to the integration algorithm (7), by applying the Laplace Transform to equation (10):

$$\frac{Q(\tau s)}{s} = \frac{\tau \sum_{k=0}^{n} b_k e^{-k\tau s}}{1 - \sum_{k=1}^{n} a_k e^{-k\tau s}} \qquad (11)$$

Equation (11) confirms the starting hypothesis of this Appendix.